(4) The E spin operator, $\boldsymbol{\sigma}_{\pm}$, is also simply related to the E chirality operator $\gamma_{5\pm}$ which in the conventional D representation is in turn simply related to the longitudinal polarization (helicity) of the particle. The relation is not an identity because of the sign $\pm$ (see Table I). This shows most clearly why any chirality invariance requirement, such as has been used in the theory of weak interaction,[3] results in opposite helicities for particles and antiparticles.

The discussion of the E representation for Dirac particles in interaction with external fields will be dealt with in a subsequent communication.

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[3] F. C. G. Sudarshan and R. E. Marshak, Proceedings of the Padua-Venice Conference on Mesons and Newly Discovered Particles, 1958; Phys. Rev. **109**, 1860 (1958). See also R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

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A Soluble Problem in Dispersion Theory*

M. L. Goldberger and S. B. Treiman
Palmer Physical Laboratory, Princeton, University Princeton, New Jersey
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The Lee model is modified by addition of a new field θ' and a weak coupling $N+\theta \rightarrow N+\theta'$, which leads to instability of the V particle: $V \rightarrow N+\theta \rightarrow N+\theta'$. The decay amplitude is calculated to lowest order in the weak coupling by dispersion relation methods. In effect we are required to study a set of simultaneous dispersion relations. The problem is completely soluble and serves to clarify the essential structure of dispersion methods. The results agree with what one obtains, more easily in the present case, by direct methods.

I. INTRODUCTION

THE Lee model[1,2] of a soluble field theory has come to play a role similar to that of, say, the harmonic oscillator in classical mechanics. Once a model is known to be soluble by simple and straightforward methods, it is not difficult to find indirect and not-so-simple methods of solution which may nevertheless be relevant and useful in other contexts. In this essentially pedagogical spirit we discuss here the dispersion relation approach to the Lee model. The original model is slightly altered however, by addition of a weak coupling which leads to instability of one of the particles of the theory. This modification provides a physical motivation for studying matrix elements which are uninteresting in the original model and thus, as is desirable, forces us to study a set of simultaneous dispersion relations.

A second reason for enlarging the Lee dynamics in this way has to do with a dispersion relation treatment of $\pi \rightarrow \mu+\nu$ decay which we undertook previously.[3] In the present case we deal again with a decay process, and it is possible to test for errors of principle in the dispersion relation approach. This is worth while, for when applied to particle decay the dispersion methods treat renormalization questions in a way which has disturbed some of our colleagues.[4] What we find in the present model is that the dispersion approach leads to the correct solution. A practical attack on more realistic particle decay problems of course requires many approximations and assumptions beyond a commitment to dispersion relations. But granted the basic analyticity assumptions, it appears that no errors of principle enter into the application of the dispersion relation methods.

The Lee model deals with N, θ, and V fields which are coupled according to the interaction $V \rightleftarrows N+\theta$. The corresponding particles N and θ are stable; and with a suitable choice of parameters a stable V particle also exists.[5] Let the respective masses be m_N, μ, and m_V, where $m_V < m_N+\mu$. We now introduce an additional field θ', corresponding to a particle of mass μ', where $m_N+\mu' < m_V < m_N+\mu$. We also introduce a direct *weak* interaction $N+\theta \rightleftarrows N+\theta'$, which we always treat to lowest order. As a consequence of this interaction the V particle becomes unstable, decaying into $N+\theta'$ through the sequence $V \rightarrow N+\theta \rightarrow N+\theta'$. Our problem is to calculate the decay amplitude—to first order in

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[5] V. Glaser and G. Källen, Nuclear Phys. **2**, 706 (1957).

the weak coupling but to all orders in the strong coupling. The calculation is a trivial one when direct methods are employed. Our purpose, however, is to approach the problem with the elaborate machinery of the dispersion relations.

Notice that the model we deal with has been constructed in analogy with charged pion decay, which we suppose proceeds mainly through the sequence: pion $\rightarrow$ baryon pairs $\rightarrow$ leptons.

II. MODEL

The Lee model has been extensively studied and there is no need here to repeat any of the discussion. We shall merely rewrite the expression for the total Hamiltonian, including the additional terms describing the θ' particle and the weak interaction $N+\theta\rightleftarrows N+\theta'$ which we are appending to the standard Lee model. We suppose that the interaction $V\rightleftarrows N+\theta$ contains a source function [$u(\omega)$ below] with properties such that all integrals which we encounter converge and such that the Lee model contains no ghost V-particle state. For simplicity we neglect recoil of the N and V particles. The Hamiltonian is

$$H=H_0+H_1+H_2, \tag{1}$$

$$H_0=m_VZ\psi_V{}^\dagger\psi_V+m_N\psi_N{}^\dagger\psi_N+\textstyle\sum_k \omega_k a_k{}^\dagger a_k +\sum_k W_k\alpha_k{}^\dagger\alpha_k, \tag{2}$$

$$H_1=g\psi_N{}^\dagger\psi_VA^\dagger+g\psi_V{}^\dagger\psi_NA+\delta m_VZ\psi_V{}^\dagger\psi_V, \tag{3}$$

$$H_2=\frac{G}{M}\psi_N{}^\dagger\psi_N(A^\dagger\mathcal{A}+\mathcal{A}^\dagger A); \tag{4}$$

where

$$A=\sum_k\frac{u(\omega)}{(2\omega\Omega)^{\frac12}}a_k,\qquad \omega=(\mu^2+k^2)^{\frac12}, \tag{5}$$

$$\mathcal{A}=\sum_k\frac{U(W)}{(2W\Omega)^{\frac12}}\alpha_k,\quad W=(\mu'^2+k^2)^{\frac12}; \tag{6}$$

and the commutation and anticommutation relations are

$$\begin{aligned}&[a_k,a_{k'}{}^\dagger]=\delta_{kk'};\quad [\alpha_k,\alpha_{k'}{}^\dagger]=\delta_{kk'},\\ &\{\psi_N,\psi_N{}^\dagger\}=1;\qquad \{\psi_V,\psi_V{}^\dagger\}=1/Z,\end{aligned} \tag{7}$$

with

$$[a_k,a_{k'}]=[\alpha_k,\alpha_{k'}]=\{\psi_N,\psi_N\}=\{\psi_V,\psi_V\}=0.$$

In Eq. (4), G is the weak coupling constant and M is a mass, inserted for dimensional reasons. We are quantizing in a box of volume Ω, where later $\Omega\rightarrow\infty$. The factor Z is a renormalization constant; g is the strong, renormalized coupling constant; and ψ_V is the renormalized V-particle field. In the absence of H_2 there is a stable V-particle state $|V\rangle$, and Z has been chosen so that

$$\langle 0|\psi_V|V\rangle=1, \tag{8}$$

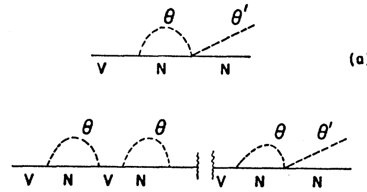

FIG. 1. Diagrams for the process $V\rightarrow N+\theta'$. Diagrams like (*b*) are absorbed in (*a*) by wave-function renormalization.

where $|0\rangle$ is the vacuum state. Finally δm_V is a V-particle mass renormalization.

From (8) and the eigenvalue problem $(H_0+H_1)|V\rangle=m_V|V\rangle$, one finds

$$\delta m_V=-\frac{g^2}{Z}\sum_k\frac{u^2(\omega)}{2\omega\Omega}\frac{1}{m_V-m_N-\omega}, \tag{9}$$

$$Z=1-g^2\sum_k\frac{u^2(\omega)}{2\omega\Omega}\frac{1}{(m_V-m_N-\omega)^2}. \tag{10}$$

One also finds readily the various states of (H_0+H_1); in particular, we are concerned with the V-particle state $|V\rangle$ and the $N+\theta$ scattering states, denoted by $|N\theta_\omega\rangle$, where ω is the energy of the θ particle. As a convention, we shall always imply by this symbol an "in"-scattering state. The state $|V\rangle$ plus the "in"-scattering states form a complete set [we have chosen the source function $u(\omega)$ such that $0<Z<1$, and thus there are no ghost states].

As for the $N+\theta'$ scattering states, to lowest order in G they are the same as the bare-particle states. We denote these by $|N\theta_W'\rangle$, where W is the energy of the θ' particle. To lowest order in G, the amplitude for $V\rightarrow N+\theta'$ decay is proportional to the matrix element $\langle N\theta_W'|H_2|V\rangle$, where $W=m_V-m_N$. More precisely, let us define the decay amplitude F according to

$$F=\frac{(2W\Omega)^{\frac12}}{U(W)}\langle N\theta_W'|H_2|V\rangle, \tag{11}$$

so that F does not depend explicitly on W. This is now the quantity that we want to evaluate; and in computing it by dispersion methods, we will be led to consider amplitudes also for various other processes. Of course the matrix element F can be computed directly from the known solution $|V\rangle$; or, what is the same thing, from use of "Feynman" diagrams. The sole diagram in question is in fact the lowest order one shown in Fig. 1(a), where the renormalized coupling constant g and physical V-particle mass m_V are to be used. The bubble diagrams in Fig. 1(b) are absorbed in Fig. 1(a) by wave-function renormalization. One finds the simple result

$$F=\frac{G}{M}g\sum_k\frac{u^2(\omega)}{2\omega\Omega}\frac{1}{m_V-m_N-\omega}=-\frac{Z}{g}\frac{G}{M}\delta m_V. \tag{12}$$

Before proceeding further, let us recall what is meant by the lifetime of the V particle. As is well known, one satisfactory way of dealing with unstable states is to regard them as resonances in scattering processes involving the decay products. One looks for a conventional resonance structure whose location and width provide a meaningful definition of both the energy and lifetime of the intermediate entity which we then describe as a decaying particle. In the present case a calculation of elastic $N+\theta'$ scattering can be carried out rigorously, and one finds a standard resonance structure in the immediate neighborhood of the energy $m_N+W=m_V$ (as expected, the precise location of the resonance is shifted from this point by terms of order G^2 and higher—this is the level shift associated with the decay process). The width of the resonance corresponds precisely to the V-particle lifetime defined by the amplitude (11). We shall not present this calculation but there is one amusing point about it worth noting: Both the elastic scattering amplitude $(N+\theta'\rightarrow N+\theta')$ and the absorption amplitude $(N+\theta'\rightarrow N+\theta)$ vanish when the energy m_N+W is equal to the *bare* V-particle mass.

III. DISPERSION RELATION APPROACH

In the present approach we pretend that the state functions $|V\rangle$ and $|N\theta_\omega\rangle$ are not known. But we are permitted, of course, to look at the Hamiltonian and make use of its properties, which in the present model are of course very simple. In contrast to more realistic situations, the required analyticity for dispersion relations can be established for all the amplitudes we deal with; and we know that all our dispersion integrals converge without subtractions.

A.

Recalling that the state $|N\theta_W'\rangle$ in Eq. (11) is meant to be a bare state, we have from (4) and (11)

$$F=(G/M)\langle 0|P|V\rangle, \tag{13}$$

$$P=\psi_N A. \tag{13'}$$

We may then write, following the prescription of Lehmann, Symanzik, and Zimmermann,[6]

$$F=i\frac{G}{M}\int_{-\infty}^{\infty} e^{-im_V t}\left(i\frac{d}{dt}+m_V\right)\langle 0|T(P\psi_V^\dagger(t))|0\rangle dt, \tag{14}$$

where $\psi_V^\dagger(t)$ is the Heisenberg field operator which coincides with the Schrödinger operator $\psi_V^\dagger$ at $t=0$; and $T(\quad)$ denotes a Wick time-ordered product:

$$T(P\psi_V^\dagger(t))=\{P,\psi_V^\dagger(t)\}\theta(-t)-\psi_V^\dagger(t)P, \tag{15}$$

where θ is the step function: $\theta(\tau)=1$, $\tau>0$; $\theta(\tau)=0$, $\tau<0$. The vacuum expectation value of the second term on the right side of (15) obviously vanishes, since P is a destruction and $\psi_V^\dagger$ a creation operator. Furthermore, the equal-time anticommutator $\{P,\psi_V^\dagger(0)\}$ is equal to 0. Thus, setting

$$f(t)=\left(-i\frac{d}{dt}+m_V\right)\psi_V(t), \tag{16}$$

we define a function $F(\xi)$ according to

$$F(\xi)=i\frac{G}{M}\int_{-\infty}^{\infty} e^{-i\xi t}\langle 0|\{P,f^\dagger(t)\}\theta(-t)|0\rangle dt; \tag{17}$$

and $F(\xi=m_V)$ is our required amplitude. From the Heisenberg equations of motion

$$-id\psi_V/dt=[H,\psi_V],$$

we have

$$f=-\delta m_V\psi_V-(g/Z)\psi_N A. \tag{18}$$

The function defined in (17) is evidently analytic in the upper half of the complex ξ plane. It is real, as we shall see, for real $\xi<m_N+\mu$, and can therefore be continued also to the lower half-plane in the usual manner. We then obtain the dispersion relation

$$F(m_V)=\frac{1}{\pi}\int\frac{\operatorname{Im}F(\xi)}{\xi-m_V-i\epsilon}d\xi. \tag{19}$$

$\operatorname{Im}F$ is obtained from the first term in $\theta(-t)=\frac{1}{2}+\frac{1}{2}\epsilon(-t)$; and introducing a sum over a complete set of physical states $|s\rangle$, we obtain

$$\operatorname{Im}F(\xi)=\pi\frac{G}{M}\sum_s\langle 0|P|s\rangle\langle s|f^\dagger|0\rangle\delta(\xi-E_s), \tag{20}$$

where E_s is the energy of the state $|s\rangle$ and where we have used

$$f^\dagger(t)=e^{iHt}f^\dagger(0)e^{-iHt}.$$

Since $\langle V|f^\dagger|0\rangle$ vanishes, the only states which contribute are the $N+\theta$ scattering states $|N\theta_\omega\rangle$; so that

$$\operatorname{Im}F(\xi)=\pi\frac{G}{M}\sum_k\langle 0|P|N\theta_\omega\rangle \times\langle 0|f|N\theta_\omega\rangle^*\delta(\xi-m_N-\omega). \tag{21}$$

B.

We are thus led to consider two new matrix elements. Let us start with the $V\rightleftarrows N+\theta$ "vertex function" $K(\omega)$ defined by

$$K(\omega)\equiv\frac{(2\omega\Omega)^{\frac{1}{2}}}{u(\omega)}\langle 0|f|N\theta_\omega\rangle. \tag{22}$$

Proceeding as before, and recalling that $|N\theta_\omega\rangle$ is an

[6] Lehmann, Symanzik, and Zimmermann, Nuovo cimento **1**, 205 (1955).

"in"-scattering state, we find

$$K(\omega)=\frac{(2\omega\Omega)^{\frac{1}{2}}}{u(\omega)}i\int_{-\infty}^{\infty}e^{-i\omega t}\left(i\frac{d}{dt}+\omega\right)\times\langle 0|T(fa_k^{\dagger}(t))|N\rangle dt, \quad (23)$$

where $a_k^{\dagger}(t)$ is the Heisenberg creation operator at time t: $a_k^{\dagger}(t)=e^{iHt}a_k^{\dagger}e^{-iHt}$. Set

$$j(t)=\frac{(2\omega\Omega)^{\frac{1}{2}}}{u(\omega)}\left(-i\frac{d}{dt}+\omega\right)a_k=-g\psi_N^{\dagger}\psi_V. \quad (24)$$

Then

$$K(\omega)=i\int e^{-i\omega t}\langle 0|T(fj^{\dagger}(t))|N\rangle dt-\frac{(2\omega\Omega)^{\frac{1}{2}}}{u(\omega)}\langle 0|[a_k^{\dagger}(0),f]|N\rangle; \quad (25)$$

$$K(\omega)=-\frac{g}{Z}+i\int e^{-i\omega t}\langle 0|[f,j^{\dagger}(t)]\theta(-t)|N\rangle dt. \quad (26)$$

Once again, this defines a function analytic in the upper ω plane, and we are led to the dispersion relation

$$K(\omega)=\frac{1}{\pi}\int\frac{\operatorname{Im}K(\omega')}{\omega'-\omega-i\epsilon}d\omega'-\frac{g}{Z}. \quad (27)$$

Proceeding as before, we have

$$\operatorname{Im}K(\omega)=\pi\sum_{k'}\langle 0|f|N\theta_{\omega'}\rangle\langle N\theta_{\omega'}|j^{\dagger}|N\rangle\delta(\omega-\omega')$$
$$=\pi\sum_{k'}\frac{u(\omega')}{(2\omega'\Omega)^{\frac{1}{2}}}K(\omega')\langle N|j|N\theta_{\omega'}\rangle^{*}\delta(\omega-\omega'). \quad (28)$$

The new matrix element to which we are here led is in turn related to the process of elastic $N+\theta$ scattering. More precisely, the S matrix element for $N+\theta_{\omega'}\rightarrow N+\theta_{\omega}$ is given by

$$\langle N\theta_{\omega}\text{ "out"}|N\theta_{\omega'}\text{ "in"}\rangle$$
$$=\delta_{\mathbf{kk}'}+i\int_{-\infty}^{\infty}dt\,e^{i\omega t}\langle N|j(t)|N\theta_{\omega'}\rangle\frac{u(\omega)}{(2\omega\Omega)^{\frac{1}{2}}}$$
$$=\delta_{\mathbf{kk}'}+2\pi i\delta_{\omega\omega'}\frac{u(\omega)}{(2\omega\Omega)^{\frac{1}{2}}}\langle N|j|N\theta_{\omega'}\rangle; \quad (29)$$

and

$$\pi\sum_{k'}\delta(\omega'-\omega)\frac{u(\omega)}{(2\omega\Omega)^{\frac{1}{2}}}\langle N|j|N\theta_{\omega'}\rangle=e^{i\delta}\sin\delta, \quad (30)$$

where δ is the (S-wave) phase shift for $N+\theta$ scattering. Inserting this result into (28), we find

$$\operatorname{Im}K(\omega)=\tan\delta(\omega)\operatorname{Re}K(\omega)\,\theta(\omega-\mu), \quad (31)$$

and hence (27) becomes the integral equation

$$K(\omega)=-\frac{g}{Z}+\frac{1}{\pi}\int_{\mu}^{\infty}\frac{\tan\delta(\omega')\operatorname{Re}K(\omega')}{\omega'-\omega-i\epsilon}d\omega'. \quad (32)$$

This is a standard equation; and since the integral above vanishes as $\omega\rightarrow\infty$, we have the solution[7]

$$K(\omega)=-\frac{g}{Z}\exp\left\{\frac{1}{\pi}\int_{\mu}^{\infty}\frac{\delta(\omega')}{\omega'-\omega-i\epsilon}d\omega'\right\}. \quad (33)$$

C.

To obtain the phase shift δ, we study $N+\theta$ scattering via dispersion methods. Define

$$\mathfrak{M}(\omega)=\frac{(2\omega\Omega)^{\frac{1}{2}}}{u(\omega)}\langle N|j|N\theta_{\omega}\rangle. \quad (34)$$

Then, proceeding in the standard way we find

$$\mathfrak{M}(\omega)=i\int_{-\infty}^{\infty}dt\,e^{-i\omega t}\langle N|[j,j^{\dagger}(t)]\theta(-t)|N\rangle, \quad (35)$$

which again defines a function analytic in the upper complex ω plane. We have the dispersion relation

$$\mathfrak{M}(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\operatorname{Im}\mathfrak{M}(\omega')}{\omega'-\omega-i\epsilon}d\omega', \quad (36)$$

and

$$\operatorname{Im}\mathfrak{M}(\omega)=\pi\langle N|j|V\rangle\langle N|j|V\rangle^{*}\delta(m_V-m_N-\omega)+\pi\textstyle\sum_{k'}\langle N|j|N\theta_{\omega'}\rangle\langle N|j|N\theta_{\omega'}\rangle^{*}\delta(\omega'-\omega). \quad (37)$$

From (8), (24), and (34) it follows that

$$\operatorname{Im}\mathfrak{M}(\omega)=\pi g^2\delta(m_V-m_N-\omega)+\pi\sum_{k'}\frac{u^2(\omega')}{2\omega'\Omega}|\mathfrak{M}(\omega')|^2\delta(\omega'-\omega);$$

or, carrying out the summation (integration) over k', we have

$$\operatorname{Im}\mathfrak{M}(\omega)=\pi g^2\delta(m_V-m_N-\omega)+\frac{1}{4\pi}u^2(\omega)(\omega^2-\mu^2)^{\frac{1}{2}}|\mathfrak{M}(\omega)|^2\theta(\omega-\mu). \quad (38)$$

When this result is substituted into the dispersion equation (36), we obtain a Low-type equation,[8] whose solution is obtainable in a standard way. Namely, introduce

$$h(\omega)=g^2(m_V-m_N-\omega)^{-1}[\mathfrak{M}(\omega)]^{-1}. \quad (39)$$

Clearly $h(\omega)$ is the boundary value of a function analytic in the ω plane cut from μ to ∞, just as is $\mathfrak{M}(\omega)$; but the singularity of the latter at $\omega=m_V-m_N$ is now removed: $h(m_V-m_N)=1$. If $\mathfrak{M}(\omega)$ has no zeros, then

[7] See, for example, R. Omnes, Nuovo cimento **8**, 316 (1958).
[8] F. E. Low, Phys. Rev. **97**, 1392 (1955).

$h(\omega)$ has no singularities in the cut plane and

$$h(\omega)=1+\frac{(m_V-m_N-\omega)}{\pi}\times\int_{-\infty}^{\infty}d\omega'\frac{\mathrm{Im}h(\omega')}{(\omega'-\omega-i\epsilon)(m_V-m_N-\omega')}. \quad (40)$$

But from (38) and (39) we see that

$$\mathrm{Im}h(\omega)=\frac{g^2}{m_V-m_N-\omega}\,\mathrm{Im}\left(\frac{1}{\mathfrak{M}}\right)=-\frac{g^2}{4\pi}\frac{u^2(\omega)(\omega^2-\mu^2)^{\frac{1}{2}}\theta(\omega-\mu)}{m_V-m_N-\omega}. \quad (41)$$

Thus, we find from (40)

$$h(\omega)=1-\beta(\omega); \quad (42)$$

$$\beta(\omega)=g^2(m_V-m_N-\omega)\frac{1}{4\pi^2}\times\int_{\mu}^{\infty}d\omega'\frac{(\omega'^2-\mu^2)^{\frac{1}{2}}u^2(\omega')}{(m_V-m_N-\omega')^2(\omega'-\omega-i\epsilon)}; \quad (43)$$

and

$$\mathfrak{M}(\omega)=\frac{g^2}{m_V-m_N-\omega}\frac{1}{1-\beta(\omega)}. \quad (44)$$

Finally, from (30) and (34) we see that

$$e^{i\delta}\sin\delta=\frac{g^2}{4\pi}\frac{(\omega^2-\mu^2)^{\frac{1}{2}}u^2(\omega)}{m_V-m_N-\omega}\frac{1}{1-\beta(\omega)}, \quad (45)$$

which is the result obtained by direct methods.[1,2]

The integral of Eq. (33) can now be readily evaluated (see Appendix) and we find for the vertex function $K(\omega)$ the expression

$$K(\omega)=-g/[1-\beta(\omega)]. \quad (46)$$

Notice that $\beta(m_V-m_N)=0$, so that at $\omega=m_V-m_N$ the vertex function K is just equal in magnitude to the renormalized coupling constant g.

D.

In connection with Eq. (21), there remains to consider the amplitude

$$R(\omega)=\frac{(2\omega\Omega)^{\frac{1}{2}}}{u(\omega)}\langle 0|\psi_N A|N\theta_\omega\rangle. \quad (47)$$

Notice that

$$\frac{G}{M}\frac{U(W)}{(2W\Omega)^{\frac{1}{2}}}\frac{u(\omega)}{(2\omega\Omega)^{\frac{1}{2}}}R(\omega)$$

is just the amplitude for the process $N+\theta_\omega\rightarrow N+\theta_W'$ $(W=\omega)$. Proceeding in the now familiar way, we find

$$R(\omega)=1+i\int_{-\infty}^{\infty}e^{-i\omega t}\langle 0|[\psi_N A,j^\dagger(t)]|N\rangle\theta(-t)dt, \quad (48)$$

where the first term on the right-hand side comes from an equal time commutator. We again have the dispersion equation

$$R(\omega)=1+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{Im}R(\omega')}{\omega'-\omega-i\epsilon}d\omega'; \quad (49)$$

and

$$\mathrm{Im}R(\omega)=\pi\langle 0|\psi_N A|V\rangle\langle N|j|V\rangle^*\delta(m_V-m_N-\omega)+\pi\textstyle\sum_{k'}\langle 0|\psi_N A|N\theta_{\omega'}\rangle\langle N|j|N\theta_{\omega'}\rangle^*\delta(\omega'-\omega). \quad (50)$$

But

$$\langle N|j|V\rangle=-g,$$
$$\langle 0|\psi_N A|V\rangle=(M/G)F(m_V).$$

Using these, and (30), we find

$$\mathrm{Im}R(\omega)=-\pi(gM/G)F(m_V)\delta(m_V-m_N-\omega)+\tan\delta(\omega)\,\mathrm{Re}R(\omega)\theta(\omega-\mu); \quad (51)$$

and thus, from (49),

$$R(\omega)=1-g\frac{M}{G}F(m_V)\frac{1}{m_V-m_N-\omega}+\frac{1}{\pi}\int_{\mu}^{\infty}\frac{\tan\delta(\omega')\,\mathrm{Re}R(\omega')}{\omega'-\omega-i\epsilon}d\omega'. \quad (52)$$

The solution is again obtained by standard methods. We find

$$R(\omega)=\left(1-\frac{M}{G}\frac{F(m_V)}{Z}\frac{g}{m_V-m_N-\omega}\right)\frac{Z}{1-\beta(\omega)}. \quad (53)$$

E.

In our attempt to compute the amplitude F for $V\rightarrow N+\theta'$ decay, we have been forced to consider the $V\rightleftarrows N+\theta$ vertex function $K(\omega)$, the amplitude $R(\omega)$ for $N+\theta\rightarrow N+\theta'$ processes, and the amplitude for $N+\theta$ elastic scattering. Collecting all results, Eqs. (19), (21), (22), (46), (47), and (53), we obtain finally

$$F(m_V)=(G/M)gZI_0-g^2F(m_V)I_1, \quad (54)$$

and hence

$$F(m_V)=\frac{G}{M}gZ\frac{I_0}{1+g^2I_1}, \quad (55)$$

where

$$I_0=\frac{1}{4\pi^2}\int_{\mu}^{\infty}d\omega\frac{(\omega^2-\mu^2)^{\frac{1}{2}}u^2(\omega)}{|1-\beta(\omega)|^2(m_V-m_N-\omega)}, \quad (56)$$

$$I_1=\frac{1}{4\pi^2}\int_{\mu}^{\infty}d\omega\frac{(\omega^2-\mu^2)^{\frac{1}{2}}u^2(\omega)}{|1-\beta(\omega)|^2(m_V-m_N-\omega)^2}. \quad (57)$$

These integrals are evaluated in the Appendix, where it is shown that

$$I_0=-\frac{\delta m_V}{g^2 Z},\tag{56$'$}$$

$$I_1=\frac{1}{g^2}\left(\frac{1}{Z}-1\right).\tag{57$'$}$$

We thus find that

$$F(m_V)=-\frac{Z}{g}\frac{G}{M}\delta m_V,\tag{58}$$

which is the correct answer.

IV. DISCUSSION

One of the motivations for this investigation concerns the doubts raised in the minds of several physicists by our dispersion relation treatment of $\pi\rightarrow\mu+\nu$ decay. Insofar as matters of principle are concerned, we feel that the present calculation demonstrates the validity of the dispersion approach. The disturbing feature of our earlier work was that the pion decay amplitude turned out to be a product of two factors: one of them, a more or less recognizable term resembling what one might expect from a cutoff perturbation theory; and a factor Z_3, the pion wave function renormalization constant. It was feared that this second factor arose from our having summed a string of pion propagation bubbles [analgous to those in Fig. 1(b)], which from the standpoint of perturbation theory would have been absorbed in a wave-function renormalization factor. But this fear is unfounded. In the first place, the wave function renormalization removes only a factor $Z_3^{\frac{1}{2}}$ (in our $\pi\rightarrow\mu+\nu$ discussion, the coupling constants which appear are all renormalized). Secondly, the present model exhibits the same behavior; the factor Z in Eq. (58) has the same (and correct) origin as the corresponding factor in pion decay.

Let us recall the mechanism envisaged for the pion problem: The pion forms a virtual nucleon-antinucleon pair (via a strong interaction operator, J), and the pair annihilates to produce the lepton pair (via the μ-capture interaction which involves strongly interacting nucleon fields as well as the essentially noninteracting lepton fields; the nucleon fields form an operator P). Now a strict perturbation treatment would yield an answer proportional to the unrenormalized strong-coupling constant and to the unrenormalized weak-coupling constant; there would also appear a divergent integral, which we imagine to have a cutoff.

As for the dispersion treatment, it breaks the problem into two parts. First the pair is created by the pion operator J; then the nucleon operator P effects the annihilation of the pair (into leptons). In analyzing the first step, we never allow the pair to re-form virtually into a pion. Such terms would correspond to the string of bubbles discussed above and are not admissible; they do not in fact occur in the dispersion analysis. However, and this is the critical point, when the pair annihilates via the operator P this may take place via a π meson. Thus, the unfamiliar Z factor arises from the weak vertex and appears effectively as a renormalization of the latter. The way in which this comes about, roughly speaking, is that instead of the perturbation theory value for the weak vertex, one finds in addition a term proportional to the decay amplitude itself. This may be seen in Eq. (53) of the present paper, where the second term has come from precisely such a discrete intermediate state. Furthermore, Eqs. (55) and (57$'$) show quite explicitly the $1/Z$ factor in the denominator of the expression for the decay amplitude, just as in the pion decay problem. It is not difficult to see that the damping denominator (factor like $1/Z$) is a quite general characteristic of decay problems.

APPENDIX

1.

In connection with Eq. (33), we want to evaluate the integral

$$I(\omega)=\frac{1}{\pi}\int_\mu^\infty\frac{\delta(\omega')}{\omega'-\omega-i\epsilon}d\omega',\tag{A-1}$$

where, according to (43) and (45),

$$\tan\delta(\omega)=-\mathrm{Im}[1-\beta(\omega)]/\mathrm{Re}[1-\beta(\omega)];\tag{A-2}$$

hence

$$\delta(\omega)=-\frac{1}{2i}\ln\left[\frac{1-\beta(\omega)}{1-\beta^*(\omega)}\right].\tag{A-3}$$

Consider the contour in the complex ω plane shown in Fig. 2, where the curve C_1 runs from ∞ to μ just below the real axis and then back to ∞ just above the axis. Since $\beta^*(\omega+i\epsilon)=\beta(\omega-i\epsilon)$, it is clear that

$$I=-\frac{1}{2\pi i}\int_{C_1}\ln[1-\beta(\omega')]\frac{1}{\omega'-\omega}d\omega'.\tag{A-4}$$

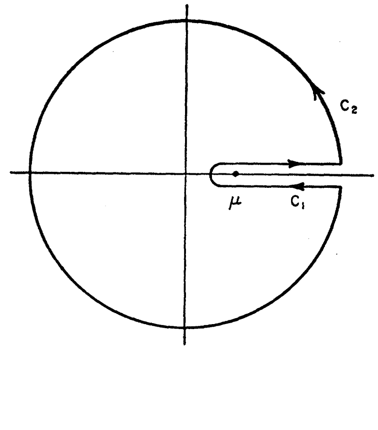

FIG. 2. Contour in complex ω plane.

Let C_2 be the contour along the infinite circle. Then

$$I=-\frac{1}{2\pi i}\left\{\oint \ln[1-\beta(\omega')]\frac{1}{\omega'-\omega}d\omega' - \int_{C_2} \ln[1-\beta(\omega')]\frac{1}{\omega'-\omega}d\omega'\right\}. \quad \text{(A-5)}$$

Now $1-\beta(\omega)$ has no zeros or poles in the cut plane; and from (10) and (43) we see that

$$1-\beta(\omega)\rightarrow Z, \quad |\omega|\rightarrow\infty. \quad \text{(A-6)}$$

Thus

$$I=-\frac{1}{2\pi i}\{2\pi i\ln[1-\beta(\omega)]-2\pi i\ln Z\}$$
$$=\ln\frac{Z}{1-\beta(\omega)}, \quad \text{(A-7)}$$

which is the result that led to (46) and (53).

2.

Next, consider the integral I_1 defined by Eq. (57). Noting that

$$\frac{1}{|1-\beta(\omega)|^2}=-\frac{1}{\mathrm{Im}(1-\beta)}\mathrm{Im}\left(\frac{1}{1-\beta}\right), \quad \text{(A-8)}$$

and that

$$\mathrm{Im}(1-\beta)=-g^2\frac{u^2(\omega)}{4\pi}\frac{(\omega^2-\mu^2)^{\frac{1}{2}}}{m_V-m_N-\omega}, \quad \text{(A-9)}$$

we find

$$I_1=\frac{1}{\pi g^2}\int_\mu^\infty d\omega\,\mathrm{Im}\left(\frac{1}{1-\beta(\omega)}\right)\frac{1}{m_V-m_N-\omega}. \quad \text{(A-10)}$$

Refer now to the contour of Fig. 2. Since $1-\beta(\omega+i\epsilon) =1-\beta^*(\omega-i\epsilon)$, it is clear that

$$2\pi i g^2 I_1=\int_{C_1} d\omega\frac{1}{1-\beta(\omega)}\frac{1}{m_V-m_N-\omega}$$
$$=\left(\oint-\int_{C_2}\right)d\omega\frac{1}{1-\beta(\omega)}\frac{1}{m_V-m_N-\omega}. \quad \text{(A-11)}$$

Once again, these integrals are readily evaluated. Recalling (A-6) and the fact that $\beta(m_V-m_N)=0$, we find

$$I_1=\frac{1}{g^2}\left(\frac{1}{Z}-1\right), \quad \text{(A-12)}$$

which is the result stated in (57′).

3.

To evaluate the integral I_0, defined by Eq. (56), we proceed in a similar manner, finding

$$I_0=\frac{1}{\pi g^2}\int_\mu^\infty d\omega\,\mathrm{Im}\left(\frac{1}{1-\beta(\omega)}\right). \quad \text{(A-13)}$$

Again, this can be written

$$2\pi i g^2 I_0=\left(\oint-\int_{C_2}\right)d\omega\frac{1}{1-\beta(\omega)}. \quad \text{(A-14)}$$

Since the integrand has no poles in the cut plane, the first integral vanishes. To evaluate the integral around the semicircle, we must imagine the circle to have a finite radius W, which we later allow to go to infinity. That is, we replace the integrand by its asymptotic expansion about the point at infinity:

$$\frac{1}{1-\beta(\omega)}=\left.\frac{1}{1-\beta(\omega)}\right|_{\omega\to\infty}+\left.\frac{1}{(1-\beta)^2}\frac{d\beta}{d(1/\omega)}\right|_{\omega\to\infty}\frac{1}{\omega}+\cdots$$
$$=\frac{1}{Z}+\frac{\delta m_V}{Z}\frac{1}{\omega}+\cdots, \quad \text{(A-15)}$$

where δm_V is defined in Eq. (9). The integral around the circle is now readily evaluated and we find

$$I_0=-\frac{1}{g^2}\frac{\delta m_V}{Z}, \quad \text{(A-16)}$$

which is the result stated in (56′).